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Noncommutative Geometry: Calculation of the Standard Model Lagrangian

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Abstract

The calculation of the standard model Lagrangian of classical field theory within the framework of noncommutative geometry is sketched using a variant with 18 parameters. Improvements compared with the traditional formulation are contrasted with remaining deviations from the requirements of physics.

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1 Introduction

The classical standard model Lagrangian (smL) is traditionally viewed as the sum of five terms. Schematically:

$$\mathcal{L} \;\; = \;\; \mathcal{L}^{\text{Yang-Mills}} + \mathcal{L}^{\text{Dirac}} + \mathcal{L}^{\text{Yukawa}} + \mathcal{L}^{\text{Higgs, kinetic}} + \mathcal{L}^{\text{Higgs, potential}}. \qquad (1)$$

A generalization of the first two terms in the spirit of noncommutative geometry (ncg) suffices to reproduce the complete Lagrangian. As a consequence, the Higgs sector need not be introduced "by hand", but is a result of the calculation, also acquiring a geometrical interpretation.
The principal idea was first outlined in [1] and [2], where mainly the electroweak part is dealt with. The inclusion of color followed in [3]. Illustrative calculations and a discussion of the model can e.g. be found in [4], [6], [7] and [9]. For a recent list of references also comprising related work, see [8]. This letter is based on parts of [5].

2 Mathematical Tools

Ncg is known to establish a generalization of differential geometry. In the case of the standard model, an analogue of the exterior algebra of differential forms which can be associated to a not necessarily commutative algebra is relevant. The description of its construction is the main goal of this section.
The rôle of a Riemannian manifold is taken by the "spectral triple". It consists of a unital, associative $*$-algebra $\mathcal{A}$, a Hilbert space $\mathcal{H}$ carrying a faithful $*$-representation π of $\mathcal{A}$ and a Dirac operator $\mathcal{D}$ on $\mathcal{H}$. $\mathcal{D}$ has to satisfy certain formal requirements which generalize properties of the usual Dirac operator in four dimensions and guarantee that the generalized actions are well-defined ([2], p. 541).
The first step towards the construction of the generalized differential algebra is the definition of the universal differential algebra $\Omega(\mathcal{A})$ of $\mathcal{A}$. It is a graded algebra

$$\Omega(\mathcal{A}) \;\; := \;\; \bigoplus_{k \in \mathbf{N}_0} \Omega^k(\mathcal{A})$$

consisting in each degree of finite sums of the form

$$\sum_{i,\text{ finite}} a_i^{(0)} \delta a_i^{(1)} \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} \delta a_i^{(k)}, \qquad a_i^{(l)} \in \mathcal{A},$$

where the derivation

$$\delta \;\; : \;\; \mathcal{A} \to \mathcal{A} \otimes_{\mathbf{K}} \mathcal{A}$$

maps $a \in \mathcal{A}$ on

$$\delta a \;\; := \;\; 1 \otimes_{\mathbf{K}} a - a \otimes_{\mathbf{K}} 1.$$

δ can be uniquely extended to an antiderivation in $\Omega(\mathcal{A})$. $\Omega(\mathcal{A})$ enjoys the following universal property: Let $\mathcal{F}$ be an associative, unital $*$-algebra with $\mathcal{A}$-module structure and

$$\Delta \;\; : \;\; \mathcal{A} \to \mathcal{F}$$

a derivation, then there is one and only one $\mathcal{A}$-algebra-homomorphism

$$\Pi \;:\; \Omega(\mathcal{A}) \to \mathcal{F},$$

satisfying

$$\Pi \circ \delta_{|\mathcal{A}} \;=\; \Delta.$$

In general, the image of $\Omega(\mathcal{A})$ under Π has lost the differential structure. Regaining it requires graded division by the differential ideal $\Pi(\mathcal{J})$, where $\mathcal{J}$ is defined as

$$\begin{aligned}
\mathcal{J} &:= \bigoplus_{k\in\mathbf{N}_0} [\ker^k \Pi + \delta \ker^{k-1} \Pi], \\
\ker^k \Pi &:= \ker \Pi \cap \Omega^k(\mathcal{A}), \\
\ker^{-1} \Pi &:= \{0\}.
\end{aligned}$$

In the resulting algebra

$$\begin{aligned}
\Omega_\Delta(\mathcal{A}) &:= \bigoplus_{k\in\mathbf{N}_0} \frac{\Pi(\Omega^k(\mathcal{A}))}{\Pi(\delta \ker^{k-1} \Pi)}, \\
\Pi \circ \delta &=: \Delta' \circ \Pi
\end{aligned}$$

defines an antiderivation Δ' which extends Δ to all of $\Omega_\Delta(\mathcal{A})$.
The generalized differential algebra associated to a spectral triple is $\Omega_{[\mathcal{D},\pi()]}(\mathcal{A})$, i.e., the rôle of Δ is taken by the commutator with the Dirac operator after application of the faithful representation π:

$$\Delta \equiv [\mathcal{D}, \pi()] \;:\; \mathcal{A} \to \mathcal{L}(\mathcal{H}) \equiv \mathcal{F}.$$

The unique $\mathcal{A}$-algebra-homomorphism is determined by

$$\Pi(\sum_{i,\text{ finite}} a_i^{(0)} \delta a_i^{(1)} \otimes_{\mathcal{A}} ... \otimes_{\mathcal{A}} \delta a_i^{(k)}) \;=\; \sum_{i,\text{ finite}} \pi(a_i^{(0)})[\mathcal{D}, \pi(a_i^{(1)})]...[\mathcal{D}, \pi(a_i^{(k)})].$$

Since Π and Δ' extend π and Δ, respectively, in the sequel, only π and Δ are used. For the requirements of physics, it suffices to determine one- and two-forms in the generalized differential algebra $\Omega_{[\mathcal{D},\pi()]}(\mathcal{A})$. The definitions of Yang–Mills and Dirac actions in ncg coincide formally with the traditional expressions. But since their arguments are generalized forms, they comprise the Higgs sector in addition to the massless part.

3 The Standard Model

In this section, it is illustrated how the physically-motivated choice of a certain spectral triple serves to reproduce the smL.
Physics dictates picking the algebra[1]

$$\mathcal{A} \;:=\; \mathcal{C}^\infty(\mathcal{M}, \mathbf{C} \oplus \mathbf{H} \oplus M_3(\mathbf{C}))$$

[1]The multiplication in $\mathcal{A}$ is pointwise the usual matrix multiplication.

with $\mathbf{H}$ the quaternions and $\mathcal{M}$ a four-dimensional, compact, Riemannian, $\mathcal{C}^\infty$- spin manifold. $\mathcal{A}$ reflects the gauge symmetry of the standard model, since its unitary group — apart from one $U(1)$-factor — is the required gauge group. The Hilbert space is split into a particle and an antiparticle space

$$\mathcal{H} \;:=\; \mathcal{H}_{\mathrm{p}} \oplus \mathcal{H}_{\mathrm{ap}}.$$

The particle sector consists of all known elementary fermions:

$$\begin{aligned}\mathcal{H}_{\mathrm{p}} \;:=\; & L^2(\mathcal{M},S)\otimes\{[\mathbf{C}^2_{\text{weak, left}}\otimes(\mathbf{C}^3_{\text{color, quark}}\oplus\mathbf{C}_{\text{color, lepton}})]\oplus \\ & \oplus[\mathbf{C}_{\text{weak, right}}\otimes(\mathbf{C}^3_{\text{color, quark}}\oplus\mathbf{C}^3_{\text{color, quark}}\oplus\mathbf{C}_{\text{color, lepton}})]\}\otimes \\ & \otimes\mathbf{C}^3_{\text{generations}},\end{aligned} \tag{2}$$

where S denotes the spinor bundle. The antiparticle sector is the charge conjugate of the particle space with a generalized charge conjugation

$$J := \left(\gamma^2\gamma^4\circ^-\right)\otimes J_0, \qquad J_0 := \begin{pmatrix} 0 & 1_{45} \\ 1_{45} & 0 \end{pmatrix}\circ^-. \tag{3}$$

The representation π of $\mathcal{A}$ on $\mathcal{H}$ is motivated by the symmetry properties of the particles:
For $\lambda\in\mathcal{C}^\infty(\mathcal{M},\mathbf{C})$, $q\in\mathcal{C}^\infty(\mathcal{M},\mathbf{H})$, $c\in\mathcal{C}^\infty(\mathcal{M},M_3(\mathbf{C}))$,

$$\pi(\lambda,q,c) \;:=$$

$$= \begin{pmatrix} \begin{pmatrix} q\otimes 1_9 & 0 \\ 0 & q\otimes 1_3 \end{pmatrix} & 0 & 0 & 0 \\ 0 & \begin{pmatrix} \lambda 1_9 & 0 & 0 \\ 0 & \bar\lambda 1_9 & 0 \\ 0 & 0 & \bar\lambda 1_3 \end{pmatrix} & 0 & 0 \\ 0 & 0 & \begin{pmatrix} 1_2\otimes\bar c\otimes 1_3 & 0 \\ 0 & \lambda 1_6 \end{pmatrix} & 0 \\ 0 & 0 & 0 & \begin{pmatrix} \bar c\otimes 1_3 & 0 & 0 \\ 0 & \bar c\otimes 1_3 & 0 \\ 0 & 0 & \lambda 1_3 \end{pmatrix} \end{pmatrix}. \tag{4}$$

The charge conjugation is introduced to exchange the upper-left and lower-right corners of these matrices, such that $M_3(\mathbf{C})$ also acts on particles and $\mathbf{H}$ on antiparticles.
Further, there is a generalization of the chirality operator on $\mathcal{H}$:

$$\Gamma := \gamma_5\otimes\Gamma_0, \qquad \Gamma_0 := \begin{pmatrix} -1_{24} & 0 & 0 & 0 \\ 0 & 1_{21} & 0 & 0 \\ 0 & 0 & -1_{24} & 0 \\ 0 & 0 & 0 & 1_{21} \end{pmatrix}. \tag{5}$$

The last ingredient of the spectral "triple" is the Dirac operator $\mathcal{D}$: As a combination of the usual Dirac operator associated to $\mathcal{M}$ and a finite-dimensional matrix $\mathcal{D}_0$, it is constructed according to the rule

$$\mathcal{D} \;=\; i^{-1}\gamma(d)\otimes 1_{90} + \gamma_5\otimes\mathcal{D}_0,$$

guaranteeing that $\mathcal{D}^2$ is a reasonable generalization of the Laplace operator. $\mathcal{D}_0$ has to satisfy the conditions[2]

- $\mathcal{D}_0 = \mathcal{D}_0^*$,
- $\{\mathcal{D}_0, \Gamma_0\}_+ = 0$,
- $[\mathcal{D}_0, J_0] = 0$,
- $[\mathcal{D}_0, \pi(0,0,M_3(\mathbf{C}))] = 0$,
- $[\mathcal{D}_0, Q_{\text{electromagnetic},\mathcal{H}}] = 0$.

The first three mimic properties of the usual Dirac operator in terms of finite-dimensional matrices; the fourth and fifth — most important for the Higgs mechanism — are the requirement that the parts of the algebra corresponding to unbroken symmetries commute with $\mathcal{D}_0$. These conditions determine that $\mathcal{D}_0$ is of the form

$$\mathcal{D}_0 = \begin{pmatrix} 0 & M & 0 & 0 \\ M^* & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{M} \\ 0 & 0 & M^t & 0 \end{pmatrix} \quad \text{with} \quad M = \begin{pmatrix} 1_3 \otimes m_u & 0 & 0 \\ 0 & 1_3 \otimes m_d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_e \end{pmatrix} \tag{6}$$

and 3×3 matrices $m_{u,d,e}$. Since a transformation of $\mathcal{D}_0$ by a unitary matrix u commuting with the algebra and J does not change the action functionals, the matrices $m_{u,d,e}$ can be parametrized as

$$m_u = \begin{pmatrix} m(u) & 0 & 0 \\ 0 & m(c) & 0 \\ 0 & 0 & m(t) \end{pmatrix}, \qquad m_d = W \begin{pmatrix} m(d) & 0 & 0 \\ 0 & m(s) & 0 \\ 0 & 0 & m(b) \end{pmatrix} W^*,$$
$$m_e = \begin{pmatrix} m(e) & 0 & 0 \\ 0 & m(\mu) & 0 \\ 0 & 0 & m(\tau) \end{pmatrix},$$

where $m(u)$, $m(d)$, $m(e)$ etc. refer to fermion masses and W denotes the CKM matrix. Counting the independent parameters with physical significance, one finds nine fermion masses and four relevant parameters of the CKM matrix. There is also some freedom in the choice of the scalar product on $\Omega_{[\mathcal{D},\pi()]}(\mathcal{A})$. It is given by the real part of the Dixmier trace (after multiplication by $|D|^{-4}$) — which in this case is the trace in the Clifford algebra with integration over $\mathcal{M}$ — and the finite-dimensional trace after multiplication by a matrix z, i.e. $\forall\ k \in \mathbf{N}_0$,[3]

$$\begin{aligned} \langle A, B\rangle_{\Omega_{[\mathcal{D},\pi()]}(\mathcal{A})} &:= \operatorname{Re}\left(\frac{1}{8\pi^2}\int d^4x \frac{1}{4}\operatorname{tr}_4(\operatorname{tr}_{90}(zA^*B))\right), \\ A, B &\in \Omega^k_{[\mathcal{D},\pi()]}(\mathcal{A}). \end{aligned} \tag{7}$$

To make the scalar product well defined, z has to satisfy the requirements:

- positivity,
- $[z, \mathcal{D}] = 0$,
- $[z, \pi(a)] = 0$,
- $[z, J\pi(a)J^*] = 0\ \forall\ a \in \mathcal{A}$.

Thus, z is of the form

$$z = \begin{pmatrix} S & 0 \\ 0 & \tilde{S} \end{pmatrix},$$

[2]The condition $[[\mathcal{D}_0, \pi(\mathcal{A})], J_0\pi(\mathcal{A})J_0^*] = 0$ ([3], p. 6207) is already satisfied when the others are.

[3]The representatives of elements in $\Omega^k_{[\mathcal{D},\pi()]}(\mathcal{A})$ are chosen so that they are orthogonal to $\pi(\mathcal{J}^k)$ with respect to the same scalar product, since it is also defined on $\pi(\Omega^k(\mathcal{A}))$. For $k \geq 2$, the choice is relevant.

$$S = \begin{pmatrix} \frac{x}{3}1_{18} & 0 & 0 & 0 \\ 0 & 1_2 \otimes \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{pmatrix} & 0 & 0 \\ 0 & 0 & \frac{x}{3}1_{18} & 0 \\ 0 & 0 & 0 & \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{pmatrix} \end{pmatrix}, \qquad (8)$$

$\tilde{S}$ analogously, each determined by four parameters, x, y_i, $\tilde{x}$, $\tilde{y}_i > 0$. Although z is given by eight numbers, it contributes only six parameters, since the $\tilde{y}_i$ always appear in a certain combination when $g_{1,2,3}$, m_{Higgs} and the vacuum with respect to the Higgs potential are expressed in terms of the parameters of ncg. This consideration gives 19 parameters altogether. The superfluous one is eliminated when gauge fields are defined: If one considers all self-adjoint elements of $\Omega^1_{[\mathcal{D},\pi()]}(\mathcal{A})$ as gauge fields, they will not correspond to the gauge fields of the standard model, but will give an unphysical $u(1)$-field due to the superfluous $u(1)$-part of $\mathcal{A}$. To cancel this $u(1)$-field, one has to impose an additional so-called unimodularity condition: Here[4],

$$\rho \in \Omega^1_{[\mathcal{D},\pi()]}(\mathcal{A}), \qquad \rho = \rho^*,$$

is further restricted by

$$\begin{aligned} \langle \omega, i^{-1}\mathcal{P}^{(\mathrm{p})}\rho\rangle_{\Omega_{[\mathcal{D},\pi()]}(\mathcal{A})} &= \langle \omega, i^{-1}\mathcal{P}^{(\mathrm{ap})}\rho\rangle_{\Omega_{[\mathcal{D},\pi()]}(\mathcal{A})} \\ \forall\, \omega \in \Omega^1_{[\mathcal{D},\pi()]}(\mathcal{A}) \quad &\text{satisfying} \quad [\omega, J] = 0, \end{aligned}$$

with $\mathcal{P}^{(\mathrm{p}),(\mathrm{ap})}$ projection operators on the particle and antiparticle sectors, respectively, and the matrix z subject to the constraint

$$4\tilde{x} = \sum_{i=1}^{3}(3\tilde{y}_i + y_i),$$

which reduces the number of independent parameters to 18. The ncg gauge field ρ satisfying all these conditions comprises the usual gauge fields and, in addition, a function with values in the quaternions also appears. In the context of nc differential geometry, this function can therefore be naturally interpreted as a further gauge field, whereas for the physicist it is — as su(2)-doublet — the candidate for the Higgs field. In formulae:

$$\begin{aligned} \text{ordinary gauge fields} \quad : \quad & \lambda \in \Gamma(\mathcal{M}, \textstyle\bigwedge^1(T^\#\mathcal{M}) \otimes \mathbf{C}) \text{ with } \lambda = -\bar{\lambda}, \\ & q \in \Gamma(\mathcal{M}, \textstyle\bigwedge^1(T^\#\mathcal{M}) \otimes \mathbf{H}) \text{ with } q = -q^*, \\ & c \in \Gamma(\mathcal{M}, \textstyle\bigwedge^1(T^\#\mathcal{M}) \otimes M_3(\mathbf{C})) \text{ with } c = -c^*, \mathrm{tr} c = 0, \end{aligned}$$

[4]This condition or a similar one is usually ([3], p. 6227; [9], p. 7) formulated with $z = 1_{90}$. Taking the standard scalar product on $\Omega_{[\mathcal{D},\pi()]}(\mathcal{A})$ with z as above establishes a connection between z and the hypercharges, this being responsible for the further constraint on z. The remaining $U(1)$-generator is coupled with the correct hypercharges to all fermions in the generalized Dirac action.

$$\begin{aligned}
\text{Higgs} &\;:\; \Phi \in \mathcal{C}^\infty(\mathcal{M}, \mathbf{H}), \\
\text{ncg gauge field} &\;:\; \rho = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \text{ with}
\end{aligned} \tag{9}$$

$$A = \begin{pmatrix} i^{-1}\gamma(q) \otimes \begin{pmatrix} 1_9 & 0 \\ 0 & 1_3 \end{pmatrix} & \gamma_5 \otimes \left(\left(\Phi \otimes \begin{pmatrix} 1_9 & 0 \\ 0 & 1_3 \end{pmatrix} \right) M \right) \\ \gamma_5 \otimes \left(M^* \left(\Phi^* \otimes \begin{pmatrix} 1_9 & 0 \\ 0 & 1_3 \end{pmatrix} \right) \right) & i^{-1} \begin{pmatrix} \begin{pmatrix} \gamma(\lambda) & 0 \\ 0 & \gamma(\bar\lambda) \end{pmatrix} \otimes 1_9 & 0 \\ 0 & \gamma(\bar\lambda) \otimes 1_3 \end{pmatrix} \end{pmatrix},$$

$$B = i^{-1} \begin{pmatrix} 1_2 \otimes \gamma(\bar c + \frac{1}{3}\bar\lambda 1_3) & 0 & 0 & 0 & 0 \\ 0 & \gamma(\lambda) \otimes 1_2 & 0 & 0 & 0 \\ 0 & 0 & \gamma(\bar c + \frac{1}{3}\bar\lambda 1_3) & 0 & 0 \\ 0 & 0 & 0 & \gamma(\bar c + \frac{1}{3}\bar\lambda 1_3) & 0 \\ 0 & 0 & 0 & 0 & \gamma(\lambda) \end{pmatrix} \otimes 1_3.$$

In the generalized differential algebra $\Omega_{[\mathcal{D},\pi()]}(\mathcal{A})$, the corresponding curvature is a combination of the usual curvatures or field strengths F of the gauge fields, a one-form $D_{\text{cov.}}(\Phi + \Phi_{\text{vac.}})$ which is the covariant derivative of the Higgs and a function h with no counterpart in the traditional formulation:

$$\theta(\rho) \;:=\; \Delta\rho + \rho \cdot \rho =$$

$$= \begin{pmatrix} \begin{pmatrix} \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} & U \\ U^* & \begin{pmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{pmatrix} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1_2 \otimes R_1 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 \\ 0 & 0 & R_1 & 0 & 0 \\ 0 & 0 & 0 & R_1 & 0 \\ 0 & 0 & 0 & 0 & R_3 \end{pmatrix} \end{pmatrix}, \tag{10}$$

ρ given by λ, q, c, Φ as above and[5]

$$\begin{aligned}
T_1 &= -\gamma(F(q)) \otimes 1_9 + \frac{1}{2} h(\Phi + \Phi_{\text{vac.}}) 1_6 \otimes ((m_u^2 + m_d^2) - \nu 1_3), \\
T_2 &= -\gamma(F(q)) \otimes 1_3 + \frac{1}{2} h(\Phi + \Phi_{\text{vac.}}) 1_2 \otimes (m_e^2 - \nu 1_3), \\
U &= -\gamma_5 \otimes \left(\begin{pmatrix} D_{\text{cov.}}(\Phi + \Phi_{\text{vac.}}) \otimes 1_9 & 0 \\ 0 & D_{\text{cov.}}(\Phi + \Phi_{\text{vac.}}) \otimes 1_3 \end{pmatrix} M \right),
\end{aligned}$$

[5] $\Gamma(\mathcal{M}, \bigwedge(T^\# \mathcal{M})) \cong \Omega_{i^{-1}\gamma(d())}(\mathcal{C}^\infty(\mathcal{M}))$ ([2], pp. 551-552).

$$
\begin{aligned}
V_1 &= -\gamma(F(\lambda)) \otimes 1_9 + h(\Phi + \Phi_{\text{vac.}})1_3 \otimes (m_u^2 - \mu 1_3),\\
V_2 &= \gamma(F(\lambda)) \otimes 1_9 + h(\Phi + \Phi_{\text{vac.}})1_3 \otimes (m_d^2 - \mu 1_3),\\
V_3 &= \gamma(F(\lambda)) \otimes 1_3 + h(\Phi + \Phi_{\text{vac.}})(m_e^2 - \mu 1_3),\\
R_1 &= -\gamma(F(\bar{c})) \otimes 1_3 + \gamma(F(\lambda))\frac{1}{3} \otimes 1_9,\\
R_2 &= -\gamma(F(\lambda)) \otimes 1_6 - \mu h(\Phi + \Phi_{\text{vac.}})1_6,\\
R_3 &= -\gamma(F(\lambda)) \otimes 1_3 - \mu h(\Phi + \Phi_{\text{vac.}})1_3
\end{aligned}
$$

with

$$
\begin{aligned}
&F(a) := da + a \wedge a, && \Phi_{\text{vac.}} := 1_2,\\
&h(\Psi) := \tfrac{1}{2}\mathrm{tr}(\Psi\Psi^*) - 1, && D_{\text{cov.}}(\Psi) := i^{-1}\gamma\left((d+q)(\Psi) - \Psi\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda}\end{pmatrix}\right),\\
&\mu := \frac{x tr(m_u^2 + m_d^2) + tr_{y_{1,2,3}}(m_e^2)}{6x + \sum_{i=1}^3 (y_i + 3\tilde{y}_i)}, && \nu := \frac{x tr(m_u^2 + m_d^2) + tr_{y_{1,2,3}}(m_e^2)}{3x + \sum_{i=1}^3 y_i}.
\end{aligned}
$$

To calculate the Yang–Mills action

$$
\mathcal{YM}^{(\text{ncg})}(\rho) \;:=\; -\langle \theta(\rho), \theta(\rho)\rangle_{\Omega_{[\mathcal{D},\pi()]}(\mathcal{A})},
$$

one mainly has to square the matrix $\theta(\rho)$ and take the trace with respect to z. The usual curvature terms lead to the ordinary Yang–Mills terms, the covariant derivative of the Higgs yields its kinetic term with correct gauge boson coupling, and, in addition, the function h gives rise to the Higgs potential, i.e., all bosonic parts of the smL are unified in the generalized Yang–Mills action, and the particular shape of the Higgs potential follows from the calculation.
In the case of the Dirac action

$$
\mathcal{DIRAC}^{(\text{ncg})}(\psi, \rho) \;:=\; -\langle \psi, (\mathcal{D} + \rho + J\rho J^*)\psi\rangle_{\mathcal{H}},
$$

sandwiching the generalized and with respect to the charge conjugation symmetrized[6] gauge connection between $\mathcal{H}$-elements leads to the unification of the massless fermionic terms and the Yukawa couplings.

4 Remarks

The achievement of ncg with respect to the smL is the formal unification of the Higgs field and the traditional gauge fields. At the same time, the model shows several deviations from the physical Lagrangian: The space–time manifold $\mathcal{M}$ has Euclidean signature and is compact (these shortcomings have to date only been resolved by ad hoc assumptions, thereby losing fundamental properties within the framework of ncg ([5])). Further, the ncg scheme produces four times as many fermions as required ([6]), due to the existence of discrete transformations like charge conjugation and chirality in the discrete manifold as well as in the continuous one. Thus, mirrored states have to be identified before an interpretation can start. The introduction of neutrino masses is not obvious either ([7]), since it would reduce the number of independent parameters

[6]Since all symmetry properties of each fermion are represented on its Hilbert space sector only after symmetrizing with respect to J, the generalized Dirac action has to be invariant under J.

by imposing further commutation relations on the matrix z determining the scalar product. Finally, the whole scheme is only useful at the level of a classical field theory. In view of these deviations, the main value of the method illustrated may be seen in the conceptual unification of the massless and the massive sectors of the smL. At this stage, new predictions concerning physical parameters are perhaps not easy to make.

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